Differential Equations

$$\vec{\mathbf{v}} = \frac{\mathrm{d}\vec{\mathbf{r}}}{\mathrm{d}t} \text{ velocity}$$
$$\vec{\mathbf{a}} = \frac{\mathrm{d}\vec{\mathbf{v}}}{\mathrm{d}t} = \frac{\mathrm{d}^2\vec{\mathbf{r}}}{\mathrm{d}t^2} \text{ acceleration}$$
$$\vec{\mathbf{r}}(t) = \vec{\mathbf{r}}(0) + \int_0^t \vec{\mathbf{v}}(t)\,\mathrm{d}t$$
$$\vec{\mathbf{v}}(t) = \vec{\mathbf{v}}(0) + \int_0^t \vec{\mathbf{a}}(t)\,\mathrm{d}t$$

1-D Constant Acceleration

$$x_f = x_i + v_i t + \frac{1}{2}at^2$$
$$x_f = x_i + \frac{1}{2}(v_i + v_f)t$$
$$v_f = v_i + at$$
$$v_{\text{avg}} = \frac{1}{2}\left(v_i + v_f\right)$$
$$v_f^2 = v_i^2 + 2a(x_f - x_i)$$

Polar coordinates $(r, \theta) \to (x, y)$

$$\vec{\mathbf{r}} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} = \begin{pmatrix} r\cos\omega t \\ r\sin\omega t \end{pmatrix}$$

2-D Constant Acceleration

$$\vec{\mathbf{r}}_f = \vec{\mathbf{r}}_i + \vec{\mathbf{v}}_i t + \frac{1}{2}\vec{\mathbf{a}}t^2$$
$$\vec{\mathbf{r}}_f = \vec{\mathbf{r}}_i + \frac{1}{2}(\vec{\mathbf{v}}_i + \vec{\mathbf{v}}_f)t$$
$$\vec{\mathbf{v}}_f = \vec{\mathbf{v}}_i + \vec{\mathbf{a}}t$$
$$\vec{\mathbf{v}}_{\text{avg}} = \frac{1}{2}\left(\vec{\mathbf{v}}_i + \vec{\mathbf{v}}_{\mathbf{f}}\right)$$
$$v_{xf}^2 = v_{xi}^2 + 2a_x(x_f - x_i)$$
$$v_{yf}^2 = v_{yi}^2 + 2a_y(y_f - y_i)$$

Relative Motion

$$\vec{\mathbf{v}}_{C/A} = \vec{\mathbf{v}}_{C/B} + \vec{\mathbf{v}}_{B/A}$$
$$\vec{\mathbf{v}}_{B/A} = -\vec{\mathbf{v}}_{A/B}$$

Circular Motion with a constant radius

$$\vec{\mathbf{a}}_{\text{cent}} = -\omega^2 r\,\hat{\mathbf{r}} = -\frac{v_{\text{tang}}^2}{r}\,\hat{\mathbf{r}}$$
$$\vec{\mathbf{a}}_{\text{tang}} = \frac{\mathrm{d}v_{\text{tang}}}{\mathrm{d}t}\hat{\mathbf{v}}$$
$$\omega = \frac{2\pi}{T} \text{ when } \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = 0$$
$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{v_{\text{tang}}}{r}$$

$\hat{\mathbf{v}}$ and $\hat{\mathbf{r}}$ unit vectors in the $\vec{\mathbf{v}}$ and $\vec{\mathbf{r}}$ directions respectively

$$1\,\text{rev} = 2\pi\,\text{rads}$$

Friction forces

$$F_{\text{static}} \le \mu_{\text{static}} F_{\text{Normal}}$$
$$F_{\text{kinetic}} = \mu_{\text{kinetic}} F_{\text{Normal}}$$

Newton's Laws (inertial frame)

$$\sum_{\text{all}} \vec{\mathbf{F}} = m\vec{\mathbf{a}}$$

Resistance proportional to v (in oil/water)[1]

$$\vec{\mathbf{R}} = -b\vec{\mathbf{v}} \text{ resistive force}$$
$$v_T = \frac{mg}{b} \text{ terminal velocity in free fall}$$
$$v(t) = v_T\left(1 - e^{-t/\tau}\right) \text{ with } \tau \equiv \frac{m}{b}$$

Resistance proportional to v^2 (air)[2]

$$\left|\vec{\mathbf{R}}\right| = \frac{1}{2}D\rho A v^2 \text{ resistive force}$$
$$v_T = \sqrt{\frac{2mg}{D\rho A}} \text{ terminal velocity in free fall}$$

Quadratic formula

$$\text{If } 0 = ax^2 + bx + c$$
$$\text{Then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Energy and Work

$$W = \int \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} \text{ with } \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = ab\cos\theta$$
$$\vec{\mathbf{F}}_{\text{cons}} = -\frac{\mathrm{d}U}{\mathrm{d}\vec{\mathbf{r}}} \Rightarrow F_x = -\frac{\partial U}{\partial x}\ F_y = -\frac{\partial U}{\partial y}$$
$$U_{\text{gravity}} = mgy \text{ and } U_{\text{spring}} = \frac{1}{2}kx^2$$
$$K = \frac{1}{2}mv^2 \Rightarrow \Delta K = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$
$$W_{\text{cons}} = \Delta U + \Delta K$$
$$W_{\text{non-cons}} = \Delta E_{\text{int}} \text{ e.g. friction } \mu_k F_N \Delta x$$
$$\Delta E_{\text{system}} = \Delta K + \Delta U + \Delta E_{\text{int}}$$
$$\Delta E_{\text{system}} = 0 \text{ only for isolated systems}$$

Power

$$P = \frac{\mathrm{d}E}{\mathrm{d}t} = \vec{\mathbf{F}} \cdot \vec{\mathbf{v}} \Rightarrow Fv\cos\theta$$
$$\Delta E = \int P\,\mathrm{d}t$$
$$P_{\text{output}} = \eta P_{\text{input}} \text{ with } \eta \equiv \text{efficiency}$$

[1] τ the *characteristic time* and b the proportionality constant.

[2] D a dimensionless constant, ρ the density of the medium in which the object is travelling, A the surface area of the object *seen from the direction of travel*

Linear Momentum

$$\vec{\mathbf{p}} = m\vec{\mathbf{v}}$$
$$\sum \vec{\mathbf{F}} = \frac{\mathrm{d}\vec{\mathbf{p}}}{\mathrm{d}t}$$
$$\vec{\mathbf{p}}_{\text{tot}} = \sum_j \vec{\mathbf{p}}_j$$
$$\Delta\vec{\mathbf{p}}_{\text{tot}} = 0 \quad \text{isolated system, no ext. forces}$$
$$\Delta\vec{\mathbf{p}} = \vec{\mathbf{I}} = \int \sum \vec{\mathbf{F}}\,\mathrm{d}t = \vec{\mathbf{F}}_{\text{ave}}\Delta t$$

Center of Mass

$$M = \sum_j m_j \qquad \vec{\mathbf{r}}_{\text{CM}} = \frac{1}{M}\sum_j m_j\vec{\mathbf{r}}_j$$
$$\vec{\mathbf{v}}_{\text{CM}} = \frac{1}{M}\sum_j \vec{\mathbf{p}}_j = \frac{1}{M}\vec{\mathbf{p}}_{\text{tot}} \qquad \vec{\mathbf{a}}_{\text{CM}} = \frac{1}{M}\sum_j \vec{\mathbf{F}}_{\text{ext}\,j}$$

Rocket Propulsion. M: rocket mass including fuel. v_e: speed of ejected mass velocity *relative to rocket*.

$$\vec{\mathbf{F}}_{\text{thrust}} = \vec{\mathbf{v}}_e\frac{\mathrm{d}M}{\mathrm{d}t} = -\vec{\mathbf{v}}_e\frac{\mathrm{d}m_e}{\mathrm{d}t}$$
$$\frac{\mathrm{d}m_e}{\mathrm{d}t} > 0 \quad \text{mass ejection rate}$$
$$v_f - v_i = v_e \ln\left(\frac{M_i}{M_f}\right) \quad \text{for finite burn}$$
$$F_{\text{avg. thrust}} = v_e\frac{M_i - M_f}{t_{\text{burn}}}$$

Kinematic Equations for Rotation with constant α

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t} \qquad \alpha = \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$$
$$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$$
$$\theta_f = \theta_i + \frac{1}{2}(\omega_i + \omega_f)t$$
$$\omega_f = \omega_i + \alpha t$$
$$\omega_{\text{avg}} = \frac{1}{2}\left(\omega_i + \omega_f\right)$$
$$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$$

Newton's equations for rigid bodies around a fixed axis

$$\sum \vec{\boldsymbol{\tau}} = I\vec{\boldsymbol{\alpha}}$$
$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} \Rightarrow Fr\sin\theta$$
$$K_{\text{rot}} = \frac{1}{2}I\omega^2 \qquad \Delta W = \int \tau\,\mathrm{d}\theta$$
$$P = \vec{\boldsymbol{\tau}} \cdot \vec{\boldsymbol{\omega}} \Rightarrow \tau\omega\cos\theta$$
$$I = \sum_j m_j r_j^2 \quad \Rightarrow \quad I = \int r^2\,\mathrm{d}m = \int_V r^2\rho(r)\,\mathrm{d}V$$

$$I = md^2 + I_{\text{CM}}$$
Parallel axis theorem. d: distance of rot. axis from CM. I_{CM}: Mom. of inertia around CM.

Rolling Motion for an object mass m and radius R. CM is center of mass.

$$v_{\text{CM}} = \omega R \text{ and } a_{\text{CM}} = \alpha R$$
$$K_{\text{trans.}} = \frac{1}{2}mv_{\text{CM}}^2 \quad \text{and} \quad K_{\text{rot.}} = \frac{1}{2}I_{\text{CM}}\omega^2$$
$$K_{\text{total}} = K_{\text{trans.}} + K_{\text{rot.}} \quad \text{and} \quad U_g = mgy_{\text{CM}}$$

Angular Momentum

$$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}} \quad \Rightarrow \quad \vec{\mathbf{L}}_{\text{total}} = \sum_j \vec{\mathbf{r}}_j \times \vec{\mathbf{p}}_j$$
$$\sum \vec{\boldsymbol{\tau}}_{\text{ext}} = \frac{\mathrm{d}\vec{\mathbf{L}}}{\mathrm{d}t} \quad \Rightarrow \quad \text{if } \sum \vec{\boldsymbol{\tau}}_{\text{ext}} = 0 \text{ then } \vec{\mathbf{L}}_i = \vec{\mathbf{L}}_f$$
$$\vec{\mathbf{L}} = I\vec{\omega} \quad \text{for rigid bodies}$$

Vectors

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = \left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{B}}\right|\cos\theta = A_xB_x + A_yB_y + A_zB_z$$
$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = \begin{pmatrix} A_yB_z - A_zB_y \\ A_zB_x - A_xB_z \\ A_xB_y - A_yB_x \end{pmatrix}$$
$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = -\vec{\mathbf{B}} \times \vec{\mathbf{A}} \quad \text{and} \quad \vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = \vec{\mathbf{B}} \cdot \vec{\mathbf{A}}$$

Static Equilibrium

$$\sum \vec{\boldsymbol{\tau}} = 0 \quad \text{and} \quad \sum \vec{\mathbf{F}} = 0$$

Universal Gravitation where a is the *semi-major axis* of the elliptic orbit with $2a = r_{\text{min}} + r_{\text{max}}$. $G = 6.67 \times 10^{-11}\,\text{kg}\,\text{m}^2/\text{s}^2$, $R_E = 6380\,\text{km}$, $M_E = 5.97 \times 10^{24}\,\text{kg}$.

$$\vec{\mathbf{F}}_g = -\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}} \quad \Rightarrow \quad g = \frac{GM_{\text{Earth}}}{R_{\text{Earth}}^2}$$
$$U_g = -\frac{Gm_1m_2}{r} \quad \Rightarrow \quad U_g = -\frac{GM_Em_{\text{sat}}}{R_E + h_{\text{sat}}}$$
$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} \quad \text{For a small object } m \text{ with } M \gg m$$
$$E = -\frac{GM_pm_s}{2a} \quad \text{For satellites in elliptic orbit \textbf{only} with } M_{\text{planet}} \gg m_{\text{sat}}$$
$$E = -\frac{GM_pm_s}{2r} \quad \text{For satellites in circular orbit } r \text{ with } M_{\text{planet}} \gg m_{\text{sat}}$$
$$v_{\text{escape}} = \sqrt{\frac{2GM}{R}} \quad \text{for planet mass } M\text{, radius } R$$
$$T^2 = \left(\frac{4\pi^2}{GM}\right)a^3 \quad \text{where } M \text{ is the much larger object. } a = r \text{ for circular orbits.}$$